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Quantum and classical solutions for a free particle in wedge billiards

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Abstract

We have studied the quantum and classical solutions of a particle constrained to move inside a sector circular billiard with angle θ_w and its pacman complement with angle $2\pi - \theta_w$. In these billiards rotational invariance is broken and angular momentum is no longer a conserved quantum number. The 'fractional' angular momentum quantum solutions are given in terms of Bessel functions of fractional order, with indices $\lambda_p = p\pi/\theta_w$, $p = 1,2,\dots$ for the sector and $\mu_q = q\pi/2\pi - \theta_w$, $q = 1,2\dots$ for the pacman. We derive a 'duality' relation between both fractional indices given by $\lambda_p = p\mu_q/2\mu_q - q$ and $\mu_q = q\lambda_p/2\lambda_p - p$. We find that the average of the angular momentum $\hat{L}_z$ is zero but the average of $\hat{L}_z^2$ has as eigenvalues λ_p^2 and μ_q^2. We also make a connection of some classical solutions to their quantum wave eigenfunction counterparts.

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Significant progress has been made in the last few years in understanding the connection between classical and quantum solutions for problems that show chaotic behavior. A particularly important role in this progress has been played by studies in billiards with different geometries [1–4]. The dynamics of a free particle in a circular billiard is completely integrable since energy and angular momentum are conserved. A simple change of the circular billiard to the stadium billiard immediately leads to chaotic particle dynamics [4]. The analysis of these billiards eigenvalue spectra and eigenfunctions have yielded a number of clearly defined quantum manifestations of classically chaotic Hamiltonians [1,2,4]. There are also other types of billiards that, although not being explicitly chaotic, can yield interesting novel quantum and classical behavior. A case in point considered in this paper has a free particle that moves inside a boundary defined by a wedge-shaped section of a circular billiard. In this case regular 2π-rotational invariant angular momentum is no longer conserved. However, *fractional* quantum angular momentum is well defined. It is this fractional angular momentum that makes this problem interesting. There have been other studies of modified circular billiards,

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like a wedge sector in the presence of a constant linear field that shows chaotic behavior [7], and circular chaotic billiards with a straight cut [8]. These billiards are different from the ones we study in this paper. Here we consider sector billiards with angle $\theta_s = \theta_w$ and their pacman complement of angle $\theta_p = 2\pi - \theta_w$. These two types of billiards can be considered special cases of simple wedge billiards that have corner discontinuities. For our quantum analysis it is, however, relevant to separate them this way since the angular momentum spectral properties of the sector and its corresponding pacman can be related to each other.

Our original motivation to study this problem actually came from experiments carried out in pacman-type mesoscopic billiards [5,6]. These billiards were studied in the presence of a magnetic field where the breaking of rotational invariance precludes the full rotational closing of Aharonov–Bohm loops. This significantly affects their contribution to the magnetoresistence. As a first step towards understanding the magnetic field problem, we consider in this paper the quantum and classical zero field cases. The quantum problem can be studied to some extent analytically, leading to interesting duality spectral relations. We also make a connection between the quantum and classical solutions that help to understand the quantum results.

The quantum Hamiltonian for a free particle of mass M in a wedge billiard is $\hat{H} = \hat{P}^2/2M + V(r)$, with

$$V(r) = \begin{cases} 0 & r \in D, \\ \infty & r \notin D. \end{cases}$$

Here D is the domain of the wedge billiard. The corresponding time-independent Schrödinger equation is

$$(\nabla^2 + k^2)\Psi_D = 0, \tag{1}$$

with boundary condition $\Psi_C = \Psi(r \in C) = 0$, with C the boundary of the domain D. Here $k^2 = 2ME/\hbar^2$, E is the energy and $2\pi\hbar = h$ is Planck's constant. We can immediately write the full set of solutions for the sector and pacman wavefunctions that satisfy the radial and angular boundary conditions. The general normalized sector wavefunction is then given by

$$\Psi_s = \Sigma_p^\infty \Sigma_{n>1}^\infty \frac{J_{\lambda_p}\left(\dfrac{\alpha_{\lambda_p,n}}{r_w}\rho\right)\sin\lambda_p\theta}{\dfrac{\sqrt{\theta_s}}{2} r_w J_{\lambda_{p+1}}(\alpha_{\lambda_p,n})}. \tag{2}$$

Note that this wavefunction vanishes when $\theta = 0$ and $\theta = \theta_s$, and $\rho = r_w$, with r_w the sector radius. The angular boundary condition determines the values for the indices λ_p that are

$$\lambda_p = \frac{p\pi}{\theta_s}, \quad p = 1,2,3\ldots \tag{3}$$

Consider, for example, the case where the angular momenta are the integers $\lambda_p = ap$ that correspond to the sector angles $\theta_s = \pi/a$, with a an integer. Depending on the value of a we will have a set of integers that will be a subset of the index values for the angular momenta of the full circle. For example, if we take $\theta_s = \pi/4$, we get $\lambda_p = 4,8,12,16,\ldots$ or if $\theta_s = \pi/3$ we get $\lambda_p = 3,6,9,12,15,\ldots$. If the index is even we get a subset of even integer angular momenta while for the odd case we get a subset of even and odd values for the angular momentum. We can instead chose $\theta_s = b/a\pi$, which will give $\lambda_p = pb/a$, with b and a prime numbers. In this case λ_p will generally be fractional and no full circle angular momentum values will be present. In the case where the angle is irrational, say $\theta_s = \alpha\pi$, with α irrational, the situation radically changes since $\lambda_p = p/\alpha$ has no corresponding analog in the angular momenta for the complete circle or for the rational angles.

In the pacman case we can also write the complete wavefunction as

$$\Psi_p = \Sigma_{\mu_q}^\infty \Sigma_{m>1}^\infty \frac{J_{\mu_q}\left(\dfrac{\alpha_{\mu_q,m}}{r_w}\rho\right)\sin\mu_q\theta}{\dfrac{\sqrt{2\pi-\theta_w}}{2} r_w J_{\mu_{q+1}}(\alpha_{\mu_q,m})}, \tag{4}$$

where μ_q is the order of the Bessel function and $m = 1,2,3\ldots$, with $\theta_p = 2\pi - \theta_w$, and

$$\mu_q = \frac{q\pi}{(2\pi - \theta_w)}, \quad q = 1,2,3\ldots, \tag{5}$$

which is of the same form as in the sector case. Again as in that case this wavefunction satisfies the imposed angular and radial boundary conditions. Note that if we take the pacman angle equal to $\pi/4$, or equivalently as a $7\pi/4$ wedge, then $\mu_q = 4/7, 8/7, 12/7, 16/7, \ldots$ that has even numerators as in the sector case but with a fractional angular momentum index that has no counterpart in the sector nor in the circle cases. For an angle $\theta_w = \pi/a$, with a an integer, we have $\mu_q = q/(2a-1)$, $a \geqslant 1$. Or, more generally, for $\theta_w = b/a\pi$, with b and a prime numbers, we have $\mu_q = q/(2a-b)$, and we need to have that $2a \neq b$, which is satisfied for prime numbers. This is an interesting result since these fractional angular momentum cases do not correspond to cases previously studied in group theory [9], at least not to the best of our knowledge. In the irrational case with $\theta_w = 2\pi - \alpha\pi$, we have $\mu_q = q/\alpha$, which has the same value as in the sector case. In fact, we show from geometric relations between the sector and the pacman wedges and for the same θ_w, the 'duality' relations between the corresponding Bessel function fractional angular momentum indices given by

$$\mu_q = \frac{\lambda_p}{2\lambda_p - p} q \Leftrightarrow \lambda_p = \frac{\mu_q}{2\mu_q - q} p. \qquad (6)$$

We note that, although the energy eigenvalues for the sector and the pacman given by $E_{\mu_q m} = \hbar\alpha^2_{\mu_q,m}/2M$ and $E_{\lambda_p n} = \hbar\alpha^2_{\lambda_p,n}/2M$, are not the same, this duality relation gives a nontrivial connection between fractional Bessel functions and the corresponding 'fractional' angular momenta for the sector and its pacman complement. In the wedge billiards rotational invariant angular momentum is not a good quantum number. We calculate then the average of the z-component of the angular momentum $\hat{L}_z = (\hbar/i)\partial/\partial\theta$, using the wavefunction given in Eq. (2), and we get

$$\langle \Psi_s | \hat{L}_z | \Psi_s \rangle = \left(\frac{\hbar}{i}\right) \frac{\sin^2 \lambda_p \theta_w}{\theta_w} = 0, \qquad (7)$$

and also for the pacman

$$\langle \Psi_p | \hat{L}_z | \Psi_p \rangle = \left(\frac{\hbar}{i}\right) \frac{\sin^2 \mu_q (2\pi - \theta_w)}{(2\pi - \theta_w)} = 0. \qquad (8)$$

These results can be physically understood from a semi-classical analysis. In a full circle we have that the particle motion completes a full rotation between 0 and 2π. For the wedges the angular motion is limited to be between $\theta \in [0, \theta_w]$ for the sectors or $\theta \in [\theta_w, 2\pi]$ for the pacmen. Since in the sector the particle moves from 0 to θ_w and then bounces back to move from θ_w to 0, the motion is librational rather than rotational. One result of this is that for the separable eigenfunctions given above all have zero average L_z. When the particle bounces off the outer periphery, only v_r changes and L_z is unchanged. When it bounces off either wedge wall, v_θ changes sign and thus L_z^2 is unchanged. This quantum result comes from our use of a specific coordinate system with origin of rotation at the apex of the billiard. However, in the classical analysis described below we also find internal closed orbits in the billiards where the angular momentum is not zero. To represent the latter ones we need to write a linear combination of the apex centered eigenfunctions and carry out a coordinate transformation to define the new angular momentum with center of rotation away from the apex.

Because of the v_r cancellations with the boundary collisions we are led to consider instead $\hat{L}_z^2$ that would take care of these cancellations. Using the full sector and pacman wavefunctions we get

$$\langle \Psi_s | \left(\hat{L}_z\right)^2 | \Psi_s \rangle = -\left(\frac{\hbar}{i}\right)^2 \lambda_p^2, \qquad (9)$$

and

$$\langle \Psi_p | \left(\hat{L}_z\right)^2 | \Psi_p \rangle = -\left(\frac{\hbar}{i}\right)^2 \mu_q^2. \qquad (10)$$

We then see that the square of the Bessel function indices are good quantum numbers. Note that this result applies even in the irrational θ_w angle case. Additionally, we also evaluated the expectation value of the quantum mechanical currents defined by $J \sim \mathrm{Im}[(\Psi\nabla)^*\Psi]$. We found that the angular component of the current is zero, contrary to what happens in the circle case, while the radial current component is zero in both billiard types, as in the circular case. We now make a qualitative connection of these quantum results to their classical counterparts. Consider a wedge cut from a thin disk with radius r_w. A particle moves freely within the wedge, bouncing elastically from the walls. Define a polar coordinate system (ρ, θ) with origin at the apex of the wedge billiard and with $\theta = 0$ pointing along the right hand boundary of the wedge. The particle is constrained to move in the ranges $0 \leqslant \theta \leqslant \theta_w$ and $0 < \rho < r_w$. The classical motion of the particle is complicated in polar coordinates, but the elastic collisions are simple to describe. Suppose the particle has velocity (v_ρ, v_θ) when it encounters the maximum radius of the wedge. The outer circle normal vector is perpendicular to $\hat{\theta}$, so v_θ will be unchanged. Because the collisions are elastic, the energy $m/2[v_\rho^2 + (r v_\theta)^2]$ must also be constant. This requires that $v_\rho \Rightarrow -v_\rho$. Similarly, for a collision with a wedge radial wall, $(v_\rho, v_\theta) \Rightarrow (v_\rho, -v_\theta)$. We can use dimensionless coordinates where $r_w = 1$ and $|\vec{v}| = 1$, since the geometry of the trajectory is not changed by the scaling of time and radius. In fact, the collisions with the radial walls can be removed by lifting copies of the wedge onto an infinite spiral in θ. The wedge copies are connected on the spiral in such a way that the motion of a particle The spiral is then projected onto the wedge by folding it like a fan. The mapping from the spiral to the wedge is done in two stages. First, define $\sigma \doteq \theta \bmod 2\theta_w$. Then

$$\theta = \begin{cases} \sigma, & 0 \leqslant \sigma \leqslant \theta_w \\ 2\theta_w - \sigma, & \theta_w \leqslant \sigma < 2\theta_w. \end{cases}$$

In order for an orbit to close, the particle must bounce off exactly the same point on the periphery, going in the same direction. This requires that the total angle traversed must be $b 2\theta_w$, for some positive integer b. Combining the two, a closed orbit is possible when $a\theta_c = b 2\theta_w$. It is easy to show that orbits will be closed or open independently of the starting position of the first orbit. Orbits can then be characterized by the pair of positive integers (a,b) [10]. The chord angle θ_c is related to the angle of incidence ϕ by $\theta_c = 2\phi$. In order for the chord to stay within the wedge, $\theta_c \leqslant \pi$, with $n \geqslant 2\theta_w/\pi b$. This implies, in particular, that the trajectory will be repeating if and only if θ_w and θ_c are commensurate. The particle makes a series of collisions with the outer perimeter at regular intervals $a\theta_c$.

We can now compare the spectra of two wedges to see if they both have orbits for certain common values of θ_c. In particular, consider a wedge sector with angle θ_s and its $\theta_p = 2\pi - \theta_s$ complement pacman. In order for an orbit in each to share a common θ_c, we need to have $\theta_c = b_s 2\theta_s/n_s = b_p 2\theta_p/n_p$ or $n_p/b_p = \theta_p/\theta_s\, n_s/b_s$. Thus, if the pacman complement angle is an integer multiple m of the angle of the sector, any orbit (a,b) in the wedge will have an orbit $(m * a, b)$ in the pacman for exactly the same value of θ_c.

In Fig. 1 we show an example of a pacman rational angle case with angle of $\pi/4$. In this figure we show the orbit (16,3), where the particle strikes the outer circle boundary 16 times before the orbit closes. During such a traversal, the total rotation of the particle on the lifted spiral is equal to $6 \cdot 7\pi/4$. We can also generate a whole family of classical orbits for this pacman with indices like (7,1), (8,1), (28,1), (28,3), (28,5) and so on. In Fig. 2 we show the corresponding eigenfunction for this pacman case. Note that the caustic radius and the number of small and large triangles correlate to the wavefunction amplitude densities shown in this figure. The eigenfunction calculations were carried out by solving

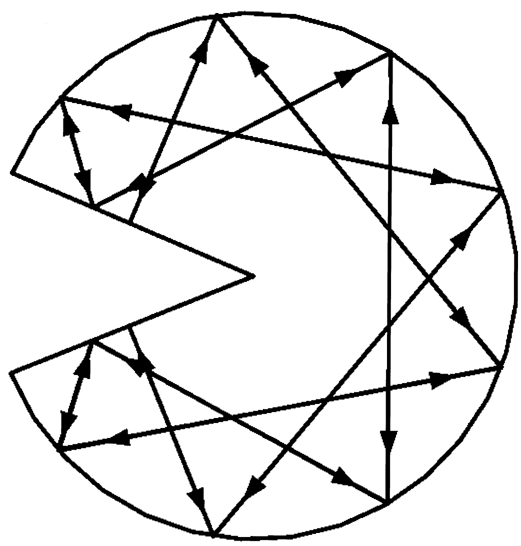

Fig. 1. Here we show the orbit (16,3) for the $\pi/4$ pacman described in the text. The particle strikes the outer circle 16 times in three rotations to close the orbit completely. The total angle covered in this orbit is equal to $6 \cdot 7\pi/4$.

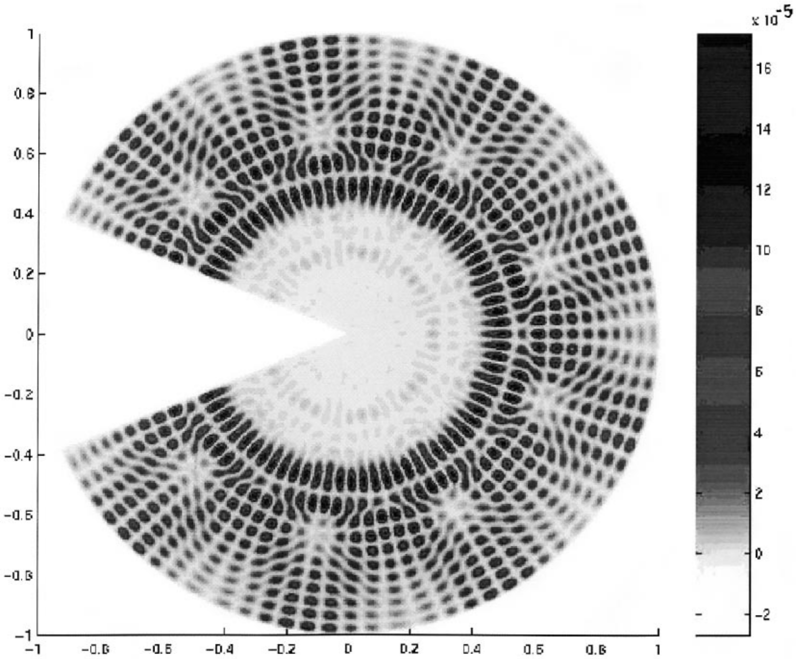

Fig. 2. In this figure we show the wavefunction density for the $\pi/4$ pacman that geometrically corresponds to the classical orbit of Fig. 1. Note that the number of small and large triangles, as well as the radius of the inner caustic, agree with the ones shown in Fig. 1.

directly the Shrödinger equation using the finite element method. In order to make a direct correlation between classical orbits and quantum eigenfunctions we need to consider all the possible allowed classical and quantum solutions for different values of the caustic radius and set of parameter orbits (a,b) and their corresponding quantum counterparts. We do not carry out this analysis since we already know the exact quantum solutions and we have described some of the classical solutions to further understand their corresponding quantum counterparts.

In conclusion, we have considered the quantum and classical problems of a free particle in wedge billiards that exhibit fractional angular momentum due to the breaking of rotational invariance. We found a new duality relation between the 'fractional' indices for the angular momentum Bessel functions for the sector and its complement pacman billiard. We showed that $\hat{L}_z^2$ becomes a good quantum operator that can be used to characterize the fractional angular momentum eigenvalues of these billiards. We will treat elsewhere the classical and quantum dynamics of a charged particle in wedge billiards in the presence of a constant homogeneous magnetic field. In that case there is a full transition from integrable to chaotic behavior as the magnetic field increases and new states of chaotic whispering gallery modes appear in the large field limit.

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