Numerical analysis of network dynamics of HH neurons

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Outline

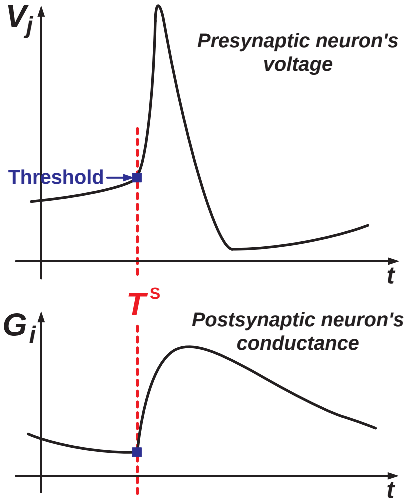
- Conductance-based Hodgkin-Huxley neurons
- Numerical results — Chaotic regimes
- Lyapunov exponents
- Summary

Conductance-based Hodgkin-Huxley neurons

- $$C_m \frac{d}{dt} V_i(t) = -G_{\mathrm{Na}} m^3 h \big(V_i(t) - E_{\mathrm{Na}}\big) - G_{\mathrm{K}} n^4 \big(V_i(t) - E_{\mathrm{K}}\big) - G_{\mathrm{L}} \big(V_i(t) - E_{\mathrm{L}}\big) - \sum_{\mathrm{Q}} G_i^{\mathrm{Q}}(t) \big(V_i(t) - E^{\mathrm{Q}}\big)$$
- $$\frac{dm}{dt} = \alpha_m(V_i)(1 - m) - \beta_m(V_i) m$$
- $$\frac{dh}{dt} = \alpha_h(V_i)(1 - h) - \beta_h(V_i) h$$
- $$\frac{dn}{dt} = \alpha_n(V_i)(1 - n) - \beta_n(V_i) n$$
- $$\frac{d}{dt} G_i^{\mathrm{Q}}(t) = -\frac{G_i^{\mathrm{Q}}(t)}{\sigma^G} + H_i^{\mathrm{Q}}(t)$$
- $$\frac{d}{dt} H_i^{\mathrm{Q}}(t) = -\frac{H_i^{\mathrm{Q}}(t)}{\sigma^H} + \sum_{j \neq i} \sum_k S_{i,j}^{\mathrm{Q}} \delta(t - T_{j,k}^S) + \sum_k F_i^{\mathrm{Q}} \delta(t - T_{i,k}^F)$$

Conductance G: voltage-threshold model

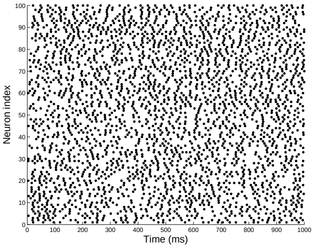
- $\frac{d}{dt}H_i^{\mathrm{Q}}(t) = -\frac{H_i^{\mathrm{Q}}(t)}{\sigma^H} + \sum_{j\neq i}\sum_k S_{i,j}^{\mathrm{Q}}\delta(t - T_{j,k}^S) + \sum_k F_i^{\mathrm{Q}}\delta(t - T_{i,k}^F)$
- $\frac{d}{dt}G_i^{\mathrm{Q}}(t) = -\frac{G_i^{\mathrm{Q}}(t)}{\sigma^G} + H_i^{\mathrm{Q}}(t)$
- $\sigma_G = 0.5\text{ms};$
- $\sigma_H = 3.0\text{ms};$
- Threshold: $V_{th} = -43\text{mV};$
- Other parameters: standard values in book [Dayan & Abbott, 2001]
- Numerical method: Runge-Kutta 4th order ODE solver with fast spike-spike correction algorithm in [Rangan & Cai, JCN 2007]

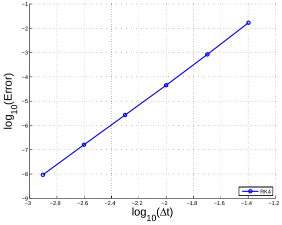


Numerical errors of different coupling strength S

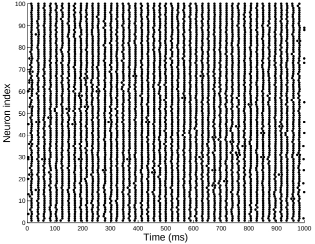
- Take $Y(t) = \left(V_1, m_1, h_1, n_1, G_1, H_1, \ldots, V_N, m_N, h_N, n_N, G_N, H_N\right)'$ with $N = 100$
- Numerical error $E(t) = \| Y^{exact}(t) - Y^{\Delta t}(t) \| = \sqrt{\sum_i \left(y_i^{exact}(t) - y_i^{\Delta t}(t)\right)^2}$

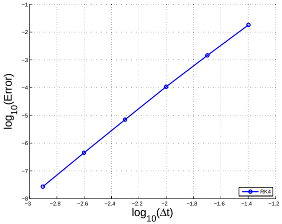
- $S = 0.2$ Asynchronous case;
- Δt : 0.04ms $\rightarrow$ 0.00125ms;
- $S = 0.8$ Synchronous case;





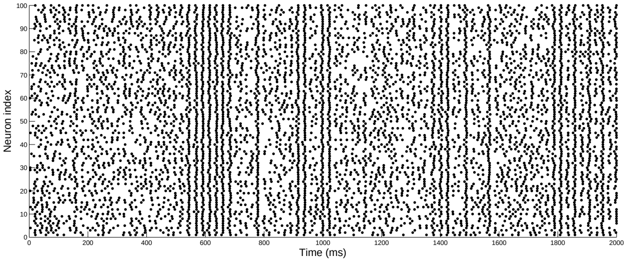
slope = 4

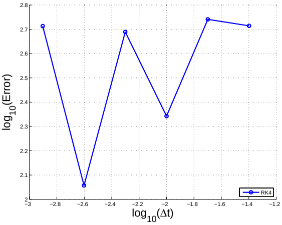




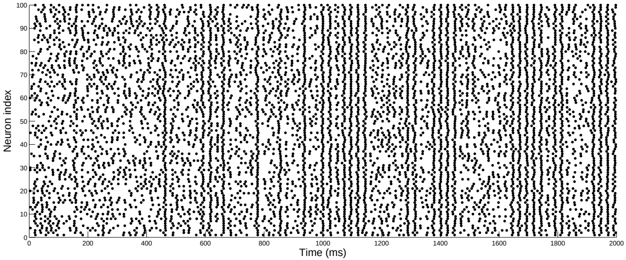
Co-existent case: $S = 0.35$: No numerical convergence!

- $\Delta t = 0.04$ms;



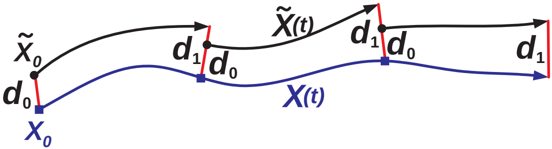


- $\Delta t = 0.02$ms;



Numerical calculation of largest Lyapunov exponent

- Examine $X(t) = \left(V_1, m_1, h_1, n_1, G_1, \ldots, V_N, m_N, h_N, n_N, G_N\right)'$, state vector of continuous variables;
- Select two very nearby points X_0 and $\tilde{X}_0$ separated by d_0 as initial conditions, (e.g d_0 =1.0e-8);
- Advance both trajectories $X(t)$ and $\tilde{X}(t)$ for time interval dt_{norm} and calculate the new separation $d_1(t)$;
- (Re-normalization): Re-adjust one trajectory so the separation is d_0 in the same direction as $d_1(t)$: $\tilde{X}(t) = X(t) + d_0 \frac{(X(t)-\tilde{X}(t))}{d_1(t)}$;

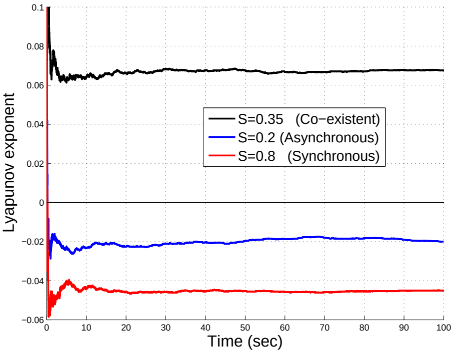


- Largest Lyapunov exponent: $\lambda = \lim_{M\to\infty} \frac{\sum_{k=1}^{M} \ln\left(\frac{d_1(t_k)}{d_0}\right)}{M\ dt_{norm}}$, where M is the total number of steps for doing re-normalization and $t_k = k\ dt_{norm}$.

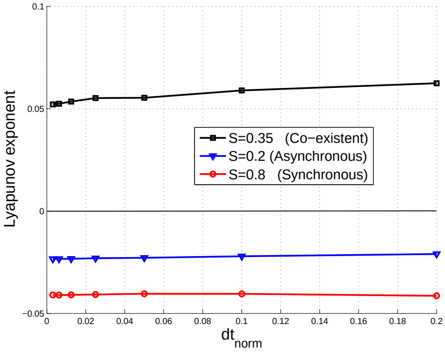
Lyapunov exponent: numerical test

- Feedforward input strength: $F = 0.1$; Input rate $r = 50$ spikes/sec.
- $S = 0.2$: Asynchronous case, $\lambda = -0.0199$;
- $S = 0.35$: Co-existent case, $\lambda = 0.0676$;
- $S = 0.8$: Synchronous case, $\lambda = -0.0450$;

Lyapunov exponent trace



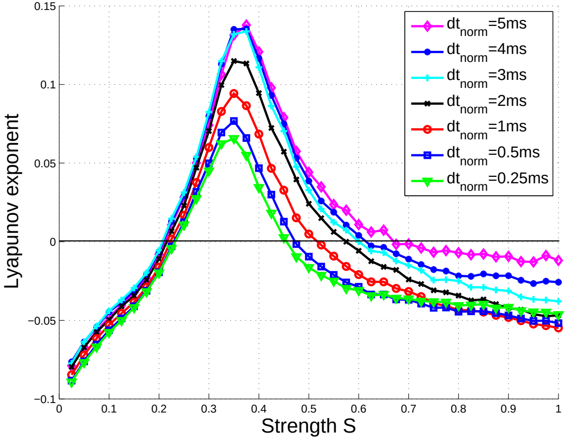
Lyapunov exponent convergence



Lyapunov exponent: numerical test

- Feedforward input strength: $F = 0.1$; Input rate $r = 50$ spikes/sec.
- Running time: $T = 100$sec; ODE step: $\Delta t = 0.04$ms.

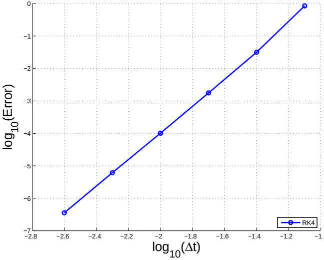
Strength S scan result



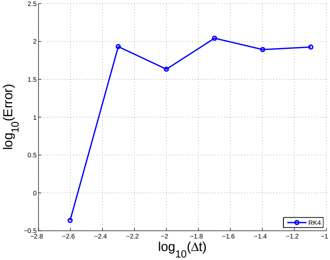
Lyapunov exponent: numerical test

- $S = 0.260$;

$T = 10$sec, $\lambda = 0.01522$

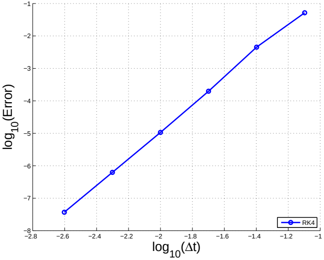


$T = 100$sec, $\lambda = 0.01589$

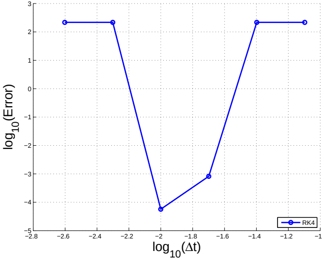


- $S = 0.456$;

$T = 10$sec, $\lambda = -0.00789$



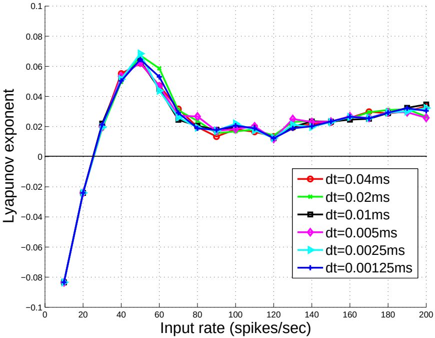
$T = 100$sec, $\lambda = -0.00216$



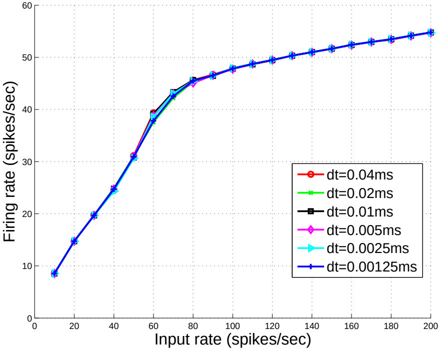
Lyapunov exponent: numerical test

- Running time: $T = 100$sec; Feedforward input strength: $F = 0.1$;
- Cortical coupling strength: $S = 0.35$ (Co-existent case);
- Scan input rate from $r = 10$ to $r = 200$ spikes/sec.
- Vary ODE step from $\Delta t = 0.04$ down to 0.00125ms; fix $dt_{norm} = 0.16$ms.

Lyapunov exponent convergence

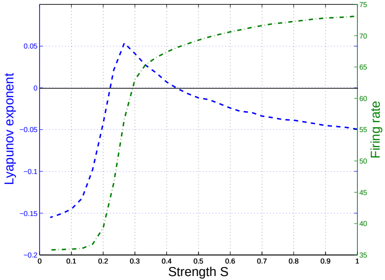


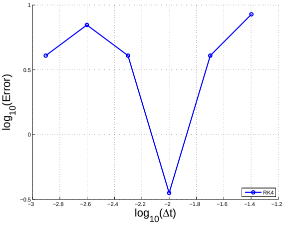
Firing rate statistically correct

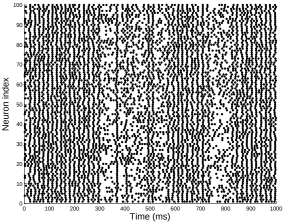


Extensions 1: conductance G_4 depends on voltage continuously

- $$\frac{d}{dt} G_{4i}(t) = -\frac{G_{4i}(t)}{\sigma} + \sum_{j \neq i} S^{E}_{i,j} g(V_j^{pre}) + \sum_k F_i^Q \delta(t - T^F_{i,k})$$
- $g(V_j^{pre}) = 1/[1 + \exp(-(V_j^{pre} - 20)/2)]$
- $$\frac{d}{dt} G_{3i}(t) = -\frac{G_{3i}(t)}{\sigma} + G_{4i}(t)$$
- $$\frac{d}{dt} G_{2i}(t) = -\frac{G_{2i}(t)}{\sigma} + G_{3i}(t)$$
- $$\frac{d}{dt} G_{1i}(t) = -\frac{G_{1i}(t)}{\sigma} + G_{2i}(t)$$
- $$\frac{d}{dt} G_{i}(t) = -\frac{G_{i}(t)}{\sigma} + G_{1i}(t)$$
- $\sigma = 0.5\text{ms}$;
- $S = 0.26$: Co-existent case, $\lambda = 0.05359$; $T = 10\text{sec}$



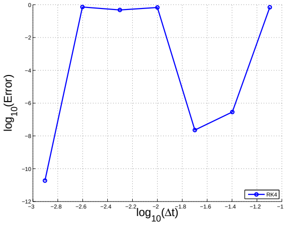


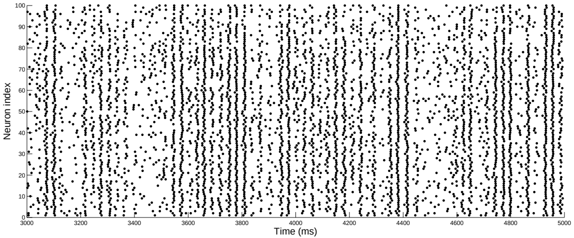


Extensions 2: add in the inhibitory neurons

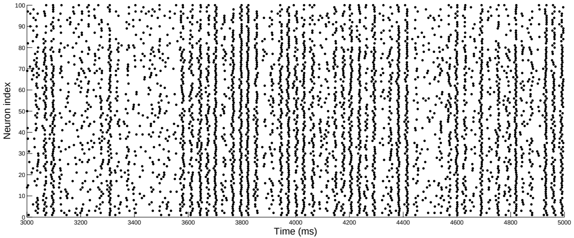
- $N^{\mathrm{E}} = 80$ excitatory neurons; $N^{\mathrm{I}} = 20$ inhibitory neurons;
- $\sigma_G^{\mathrm{E}} = 0.5$ms; $\sigma_H^{\mathrm{E}} = 3.0$ms; $\sigma_G^{\mathrm{I}} = 0.5$ms; $\sigma_H^{\mathrm{I}} = 7.0$ms;
- $S^{\mathrm{EE}} = 0.6$; $S^{\mathrm{IE}} = 1.0$; $S^{\mathrm{EI}} = 0.1$; $S^{\mathrm{II}} = 2.0$;

- $\Delta t = 0.04$ms,
  $\lambda = 0.05263$;





- $\Delta t = 0.02$ms,
  $\lambda = 0.05059$;



Summary

- In some parameter regime, we can not get numerical convergence for individual neuronal trajectories;
- These chaotic regimes may be indicated by positive largest Lyapunov exponent of the continuous system variables;
- Some statistical properties (e.g. average system firing rate) may have numerical convergence.
- On-going work: try to use some tools (return plots of system ISI, power spectra analysis of voltage traces, etc) to distinguish these regimes intuitively.

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