Synchronization in Complex Networks

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Main collaborators

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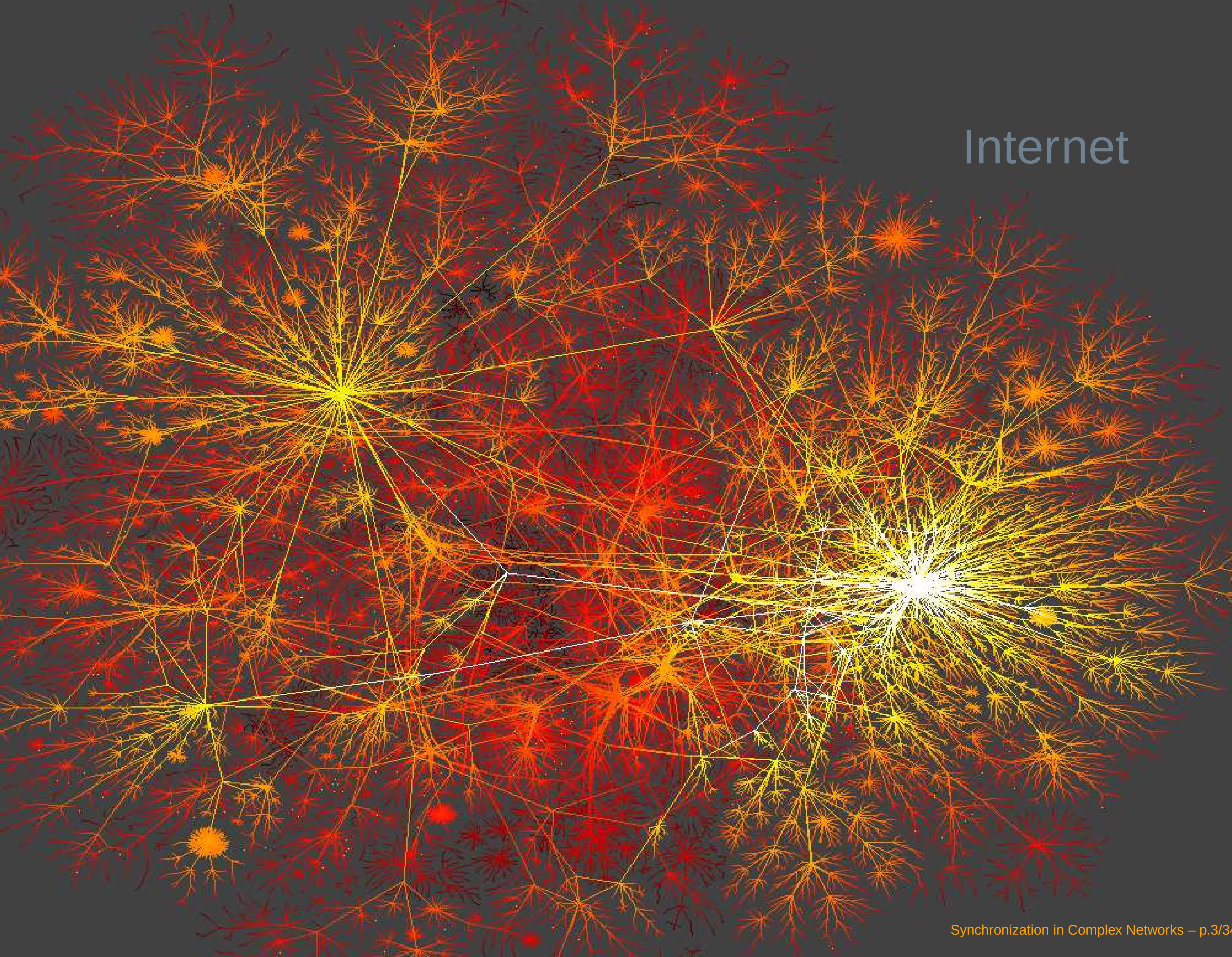
- Liang Huang, ASU
- Robert A. Gatenby, U. Arizona

Internet



Network Synchronization

- The celebrated classical Kuramoto model (1984) - globally coupled network of phase oscillators.
- Synchronization in complex networks - highly relevant to biological functions, large-scale parallel processing, coordination in technological systems, etc.

Relevance to Systems Biology

- Organizing biological information using the network idea has been fundamental to utilizing various systems-level approaches to understanding biological functions.
- Synchronizability of oscillator networks, such as the networks of cortical neurons, is thought to play an important role for the functionality of the network.
- Complex topology is common in biological networks.

Synchronization in Complex Networks

- X.-F. Wang and G. Chen, "Synchronization in small-world dynamical networks," Int. J. Bifurcation Chaos **12**, 187 (2002).
- M. Barahona and L. M. Pecora, "Synchronization in small-world systems," Phys. Rev. Lett. **89**, 054101 (2002).
- T. Nishikawa, A. Motter, Y.-C. Lai, and F. Hoppensteadt, "Heterogeneity in oscillator networks: Are smaller worlds easier to synchronize?" Phys. Rev. Lett. **91**, 014101 (2003).

Oscillator-Network Approach

$$\dot{x}_i = F(x_i) - \varepsilon \sum_{j=1}^{N} G_{ij} H(x_j)$$

G - coupling matrix, e.g.,

$$G_{ij} = \begin{cases} -1/k_i, & \text{if nodes } i \text{ and } j \text{ interact with each other} \\ 1, & \text{if } i = j \\ 0, & \text{otherwise,} \end{cases}$$

- $\sum_{j=1}^{N} G_{ij} = 0$ for all i
- How to choose G to achieve stronger network synchronizability has been an active topic of research.

Stability of Synchronous Solution

- For any solution $s(t) \in \mathbb{R}^n$ of $\dot{x} = F(x)$ (periodic, chaotic, or else), $x_i(t) = s(t)$ for all i is a synchronized solution for the whole oscillator network.
- Stability of the solution is determined by linearizing and diagonalizing the equations, after which it takes the form

$$\dot{y}_i = \big[DF(s(t)) - \varepsilon\lambda_i DH(s(t))\big] y_i, \quad {\scriptstyle i=1,\ldots,N}$$

where $\lambda_1 = 0 \leq \lambda_2 \leq \ldots \leq \lambda_N$ are the eigenvalues of L.
- Fujisaka and Yamada, 1983, 1986.

Transverse Stabilities

- $\lambda_1 = 0$ corresponds to the synchronization manifold:

$$\dot{y}_i = [DF(s(t))]y_i,$$

where DF is the Jacobian matrix of node dynamics.

- $0 \leq \lambda_2 \leq \ldots \leq \lambda_N$ correspond to $(N-1)$-dimensional transverse subspace. Let $K \equiv \varepsilon\lambda_i$ be the normalized coupling parameter. Transverse stabilities are determined by

$$\dot{y}_i = [DF(s(t)) - KDH(s(t))]y_i, \quad {\scriptstyle i=2,\ldots,N}$$

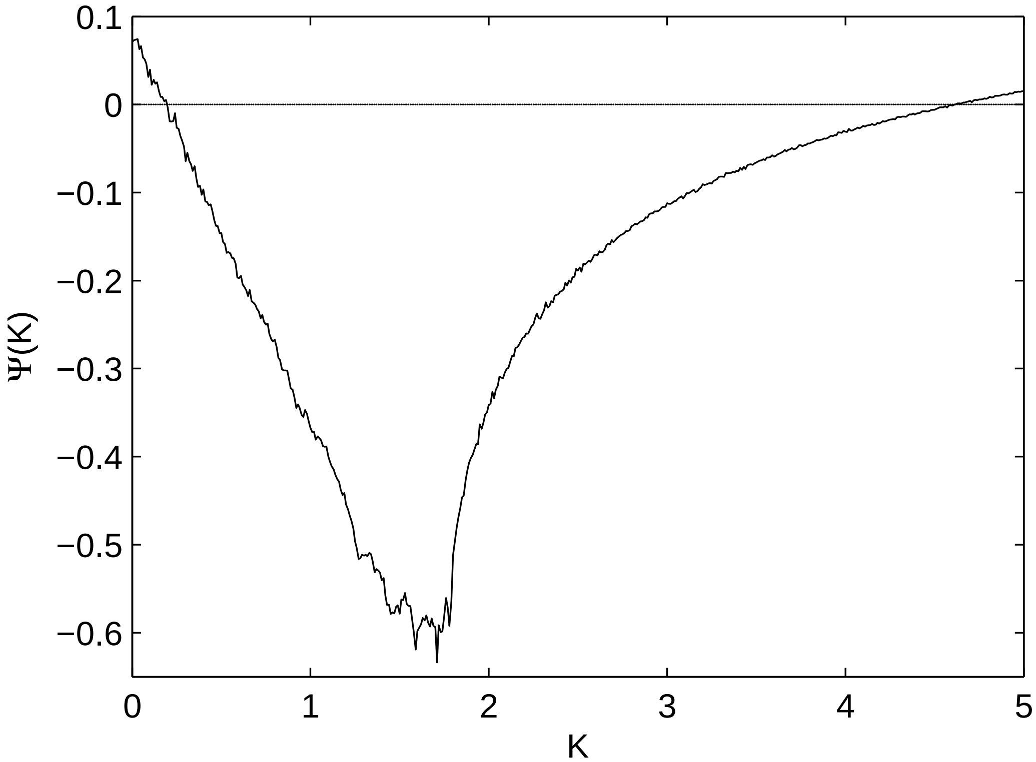
Master Stability Function

- The largest Lyapunov exponent $\Psi(K)$ (= the Floquet exponent, if periodic) for the system

$$\dot{y} = \big[DF(s(t)) - KDH(s(t))\big]y$$

  is called the master stability function. Pecora and Carroll (1998).
- Physically realizable synchronization requires $\Psi(K) < 0$.
- If $\Psi(K) < 0$ for $K_1 < K < K_2$, since $K \equiv \varepsilon\lambda_i$ ($i = 2, \ldots, N$), network synchronization occurs if $\varepsilon\lambda_2 > K_1$ and $\varepsilon\lambda_N < K_2$.

Example of Master-Stability Function



- Rössler oscillator:

  $\mathbf{F}(\mathbf{x}) = [-(y+z), x+0.2y, 0.2+z(x-9)]^T$.
- $K_1 \approx 0.2$ and $K_2 \approx 4.6$. Say $\varepsilon = 0.5$. Network synchronization requires $\lambda_2 > 0.4$ and $\lambda_N < 9.2$.

Generic MSF?

- Question raised by Lou Pecora at the September 07 Aberdeen Chaos Conference.
- Recent work at ASU: analysis and computation of MSFs for known chaotic oscillators.
- Systems studied so far: Rössler oscillator, Lorenz oscillator, Chua's circuit, Chen's oscillator, Duffing's oscillator, and van der Pol oscillator.
- For each system, examined all 9 common and experimentally implementable coupling configurations.

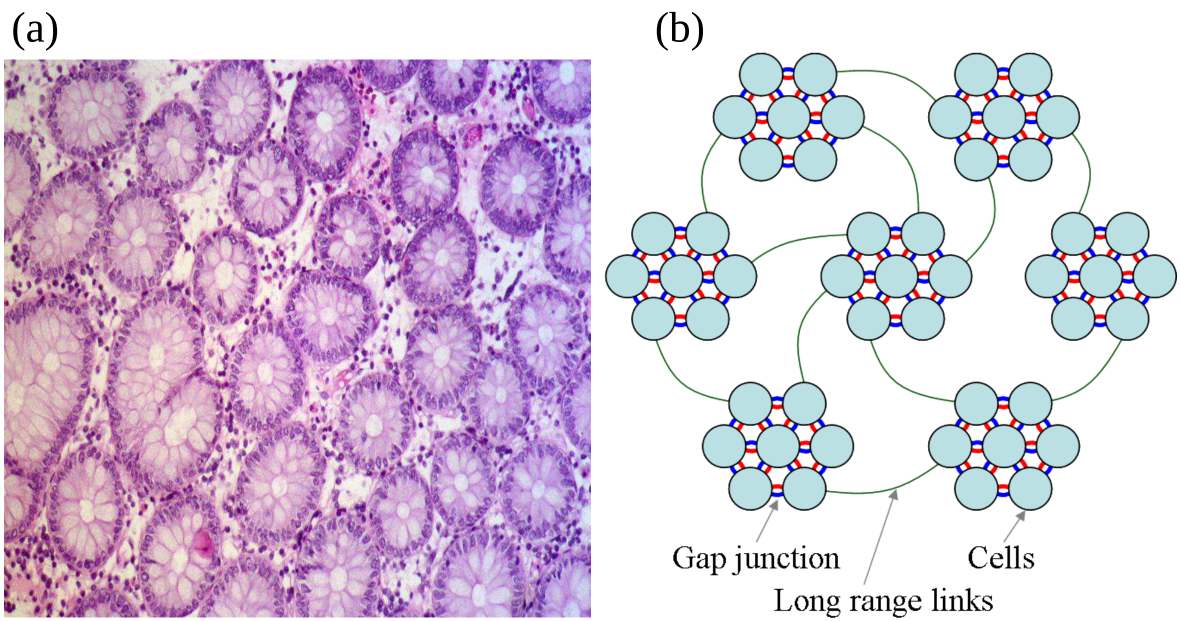
Generic MSF?

- Results: there always exists a coupling configuration for which the MSF is negative in a finite parameter interval, regardless of system details.
- L. Huang and Y.-C. Lai, “Generic master-stability function in chaotic systems,” to be published.
- L. Huang, Y.-C. Lai, and R. Gatenby, “Dynamics-based scalability in complex networks,” to be published.

Synchronization in Scale-Free Networks

- Smaller average network distance generally does not imply enhanced synchronizability.
- Heterogeneity in the degree distribution is an obstacle for synchronization. Thus, some degree of homogeneity is expected in networks for which synchronizability is essential (e.g., neuronal networks) even if network distance is not optimal.
- Heterogeneity of load distribution appears to be responsible for the counter-intuitive effect.
- T. Nishikawa, A. Motter, Y.-C. Lai, and F. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003). [202 - Google Scholar]

Cell Organization in Normal Tissue



- Microscopic image of the colon (crypts);
- Local communication subnetworks based on gap-junction conductance coupling;
- Long-range diffusive interactions based on specific cell-membrane receptors.

Complex Clustered Networks in Biology

- Metabolic networks

  E. Ravasz, A. L. Somera, D. A. Mongru, Z. Oltvai, and A.-L. Barabási, “Hierarchical organization of modularity in metabolic networks,” Science 297, 1551 (2002).

- Protein-protein interaction networks

  V. Spirin and L. A. Mirny, “Protein complexes and functional modules in molecular networks,” Proc. Natl. Acad. Sci. USA 100, 12123-12128 (2003).

Clustered Networked Systems in Engineering

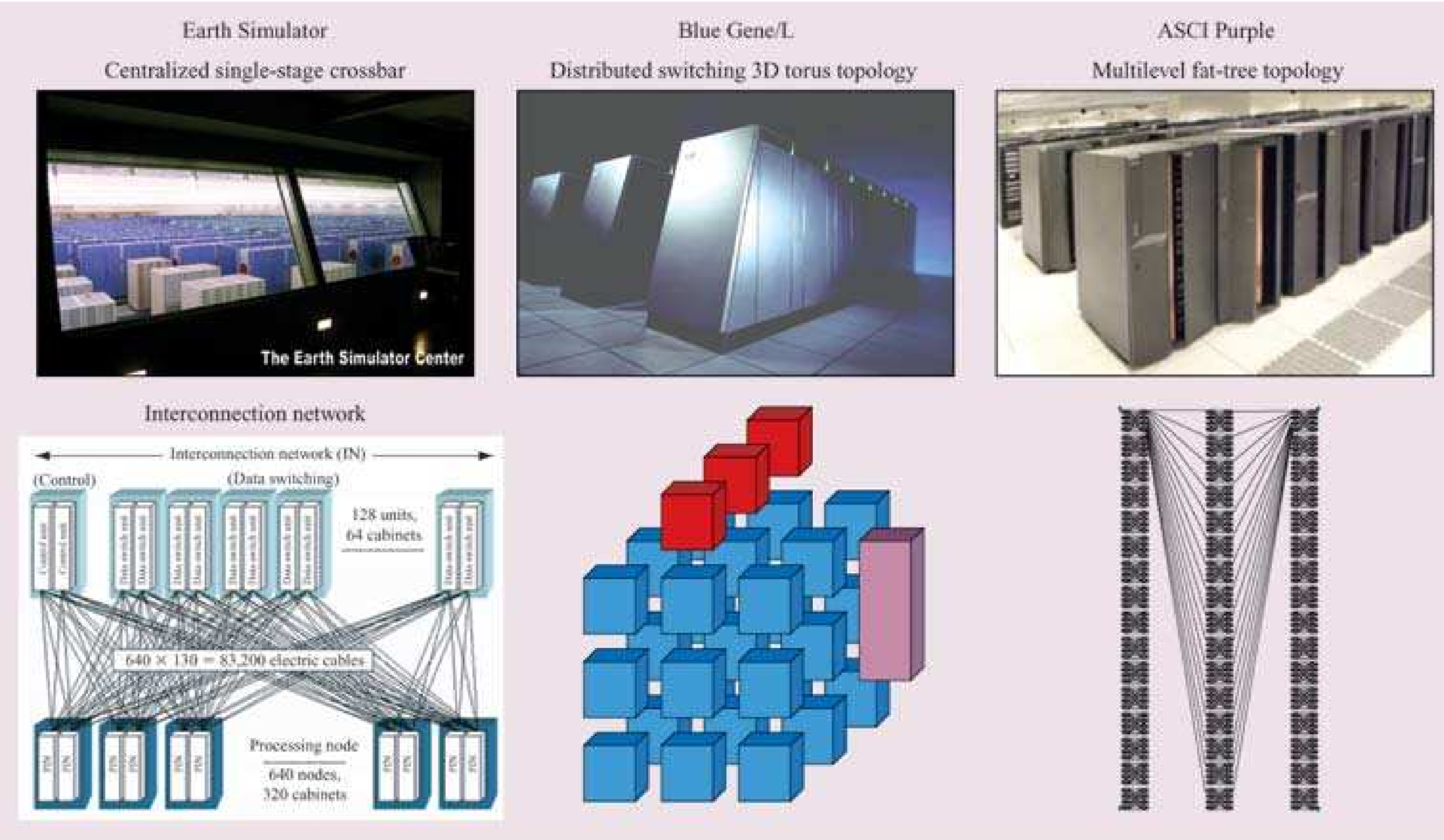
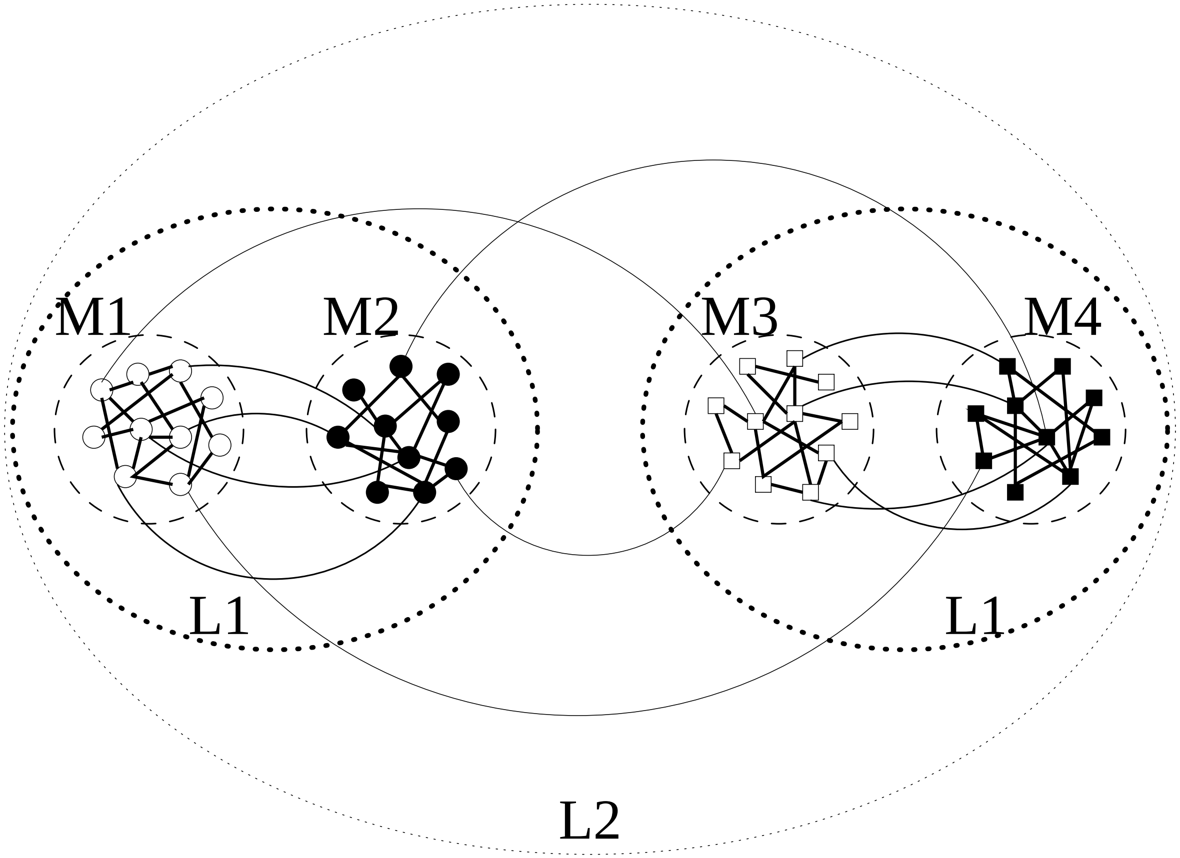


Figure 10

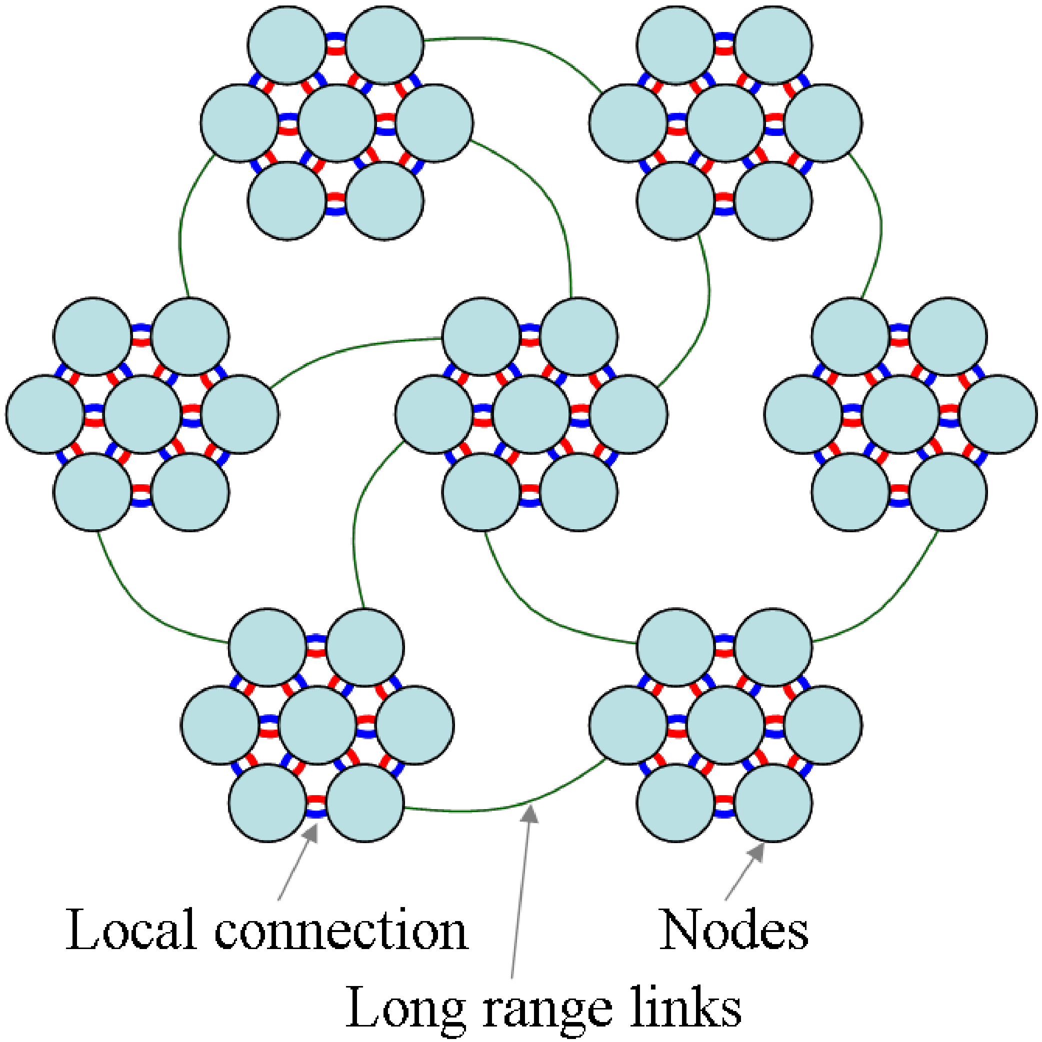
Clusters and massively parallel machines: Earth Simulator, Blue Gene/L, and ASCI Purple.

Clustered Topology in Social Networks



- W. W. Zachary, J. Anthropol. Res. **33**, 452 (1977).

Complex clustered Networks



Optimizing Synchronization

- Two key parameters of interest:
  1. p_l - probability of inter-cluster links;
  2. p_s - probability of intra-cluster links.
- In the two-dimensional parameter plane defined by $0 \leq (p_s, p_l) \leq 1$, where can the network be best synchronized?
- Intuition: as p_l or p_s is increased, there are more links in the network, so it should be easier to achieve synchronization.

Optimizing Synchronization: Main Result

- Optimal synchronizability occurs for $p_l \approx p_s$. For small p_l, increasing p_s can lead to destruction of synchronization.
- L. Huang, K. Park, Y.-C. Lai, L. Yang, and K.-Q. Yang, "Abnormal synchronization in complex clustered networks," PRL 97, 164101 (2006).
- X.-G. Wang, L. Huang, Y.-C. Lai, and C. H. Lai, "Optimization of synchronization in gradient clustered networks," PRE 6, 056113(1-5) (2007).
- L. Huang, Y.-C. Lai, and R. A. Gatenby, "Optimization of synchronization in complex clustered networks," Chaos 18, 013101(1-10) (2008).

Computational Criterion for Synchronization

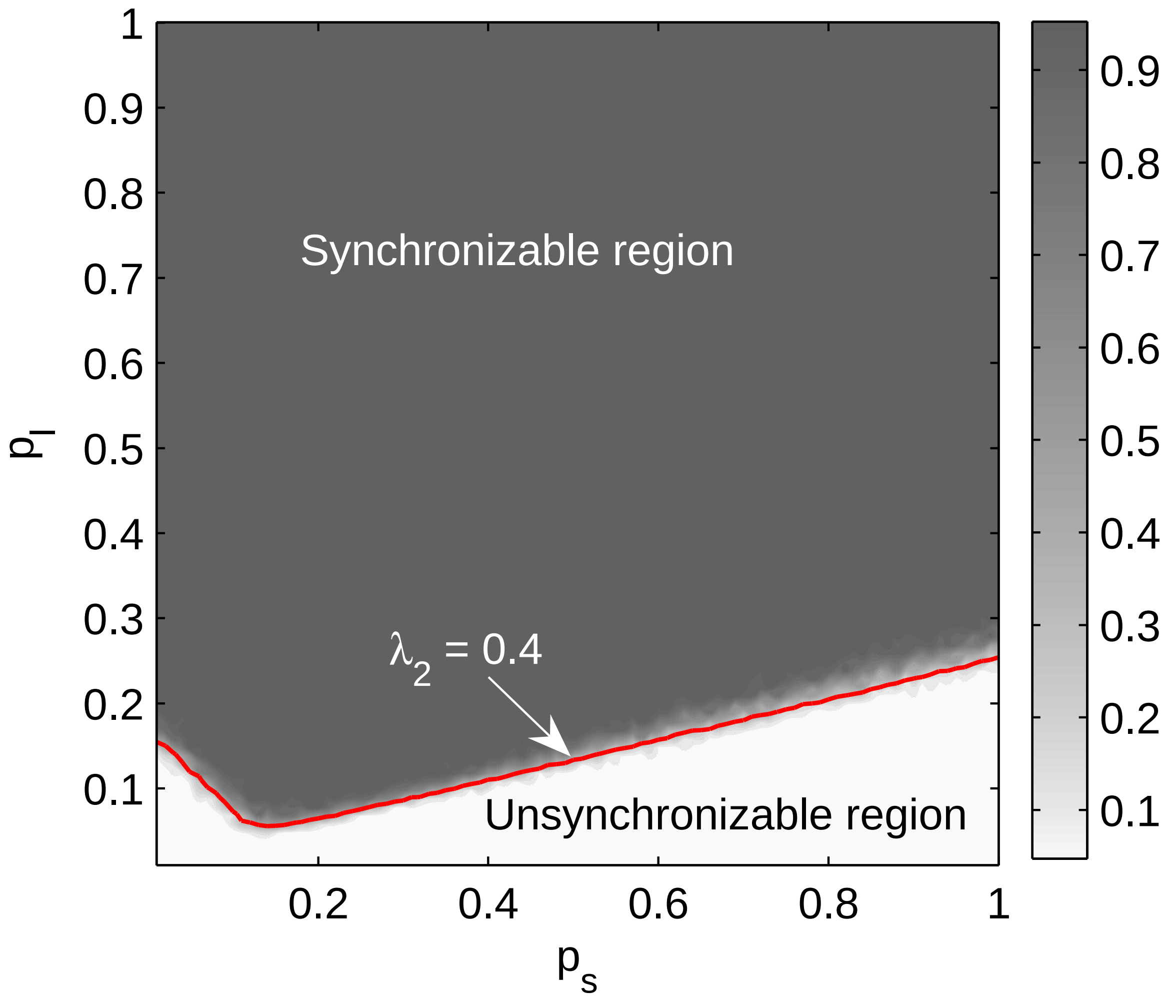
- Network synchronization is meaningful in a probabilistic sense.
- Calculate “synchronization width” defined by

$$W(t) = \langle |x(t) - \langle x(t) \rangle | \rangle,$$

where $\langle \cdot \rangle$ is average over nodes in the network.

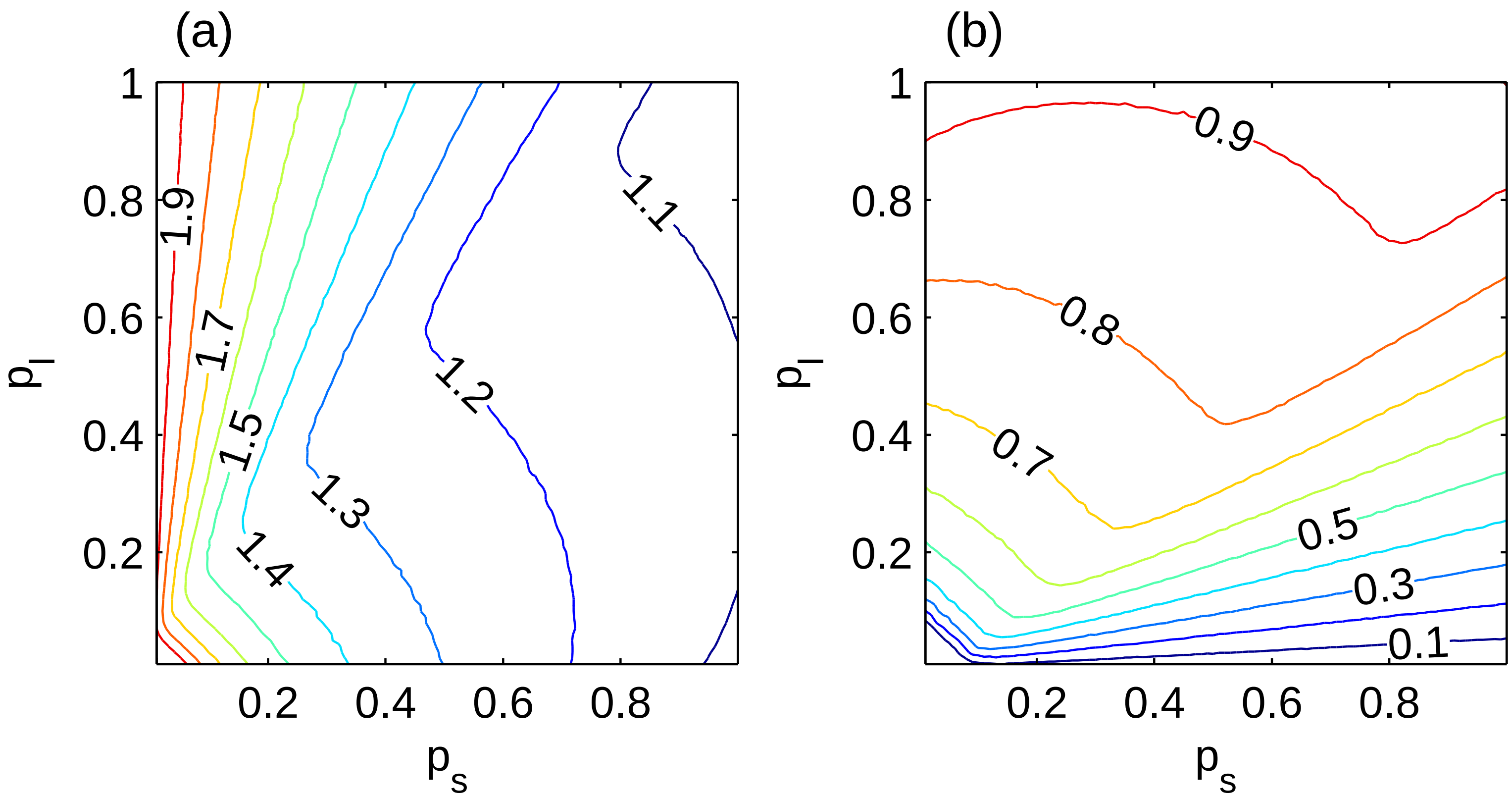
- Define P_{syn} - probability for $W(t) < \delta$ (small number, say 0.01) for a large time period.
- $P_{syn} \approx 0$ - network is unsynchronizable;
  $P_{syn} > 0$ - network is synchronizable;
  $P_{syn} \approx 1$ - network is strongly synchronizable.

Clustered Rössler Network



- Classical Rössler oscillator on each node; $N = 100$, $M = 2$, and $\varepsilon = 0.2$.

Contour Plots of λ_N and λ_2

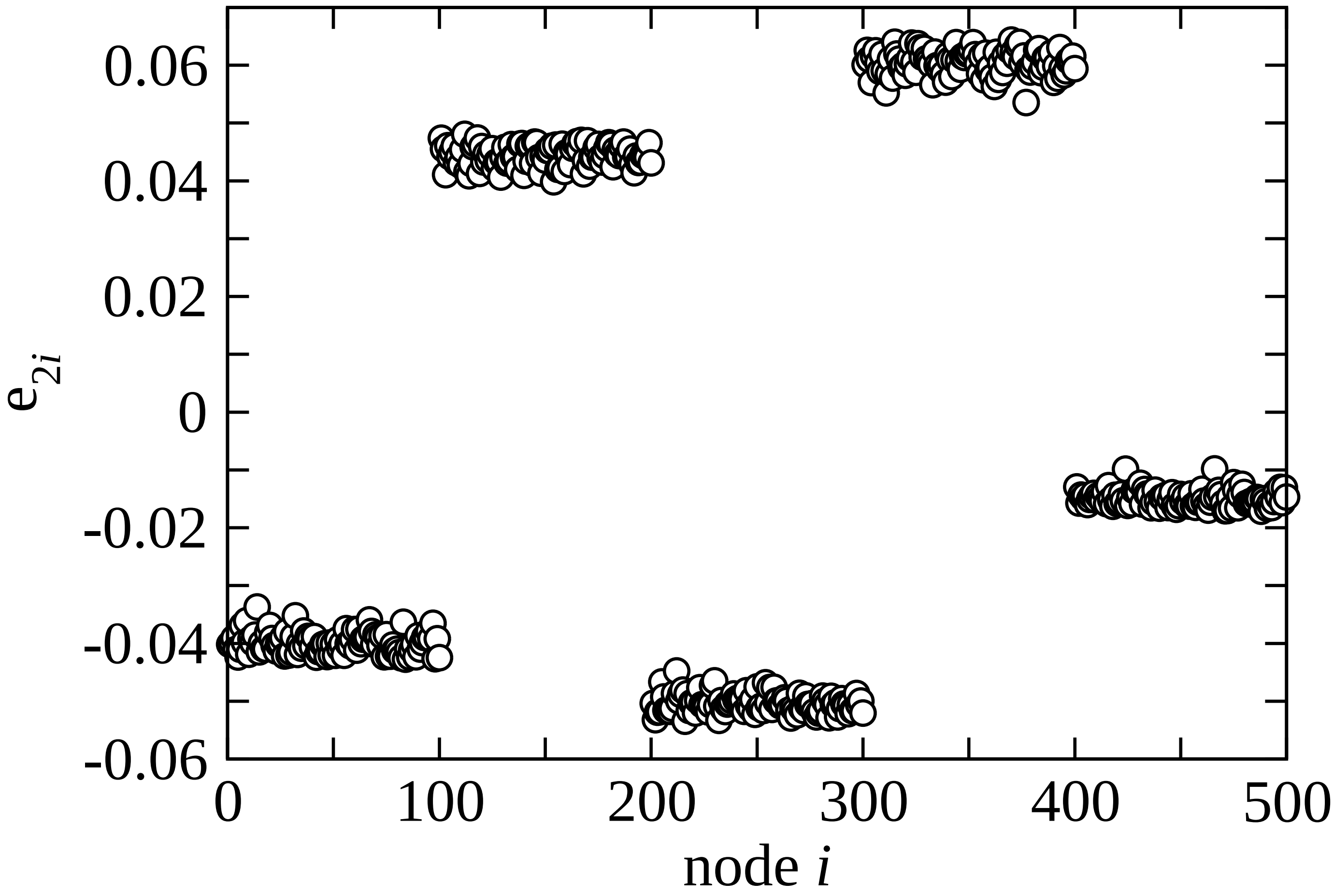


- Reminder: synchronizable if $\lambda_2 > 0.4$ and $\lambda_N < 9.2$.
- Key: λ_2. For fixed p_l (small), $\lambda_2(p_s)$ is a decreasing function for p_s large.

Theory (1)

- For a clustered network, the components of the eigenvector $\mathbf{e}_2$ have approximately the same value within any cluster, while they can be quite different for different clusters.
- So write $\mathbf{e}_2 \approx [\widetilde{e}_1, \cdots, \widetilde{e}_1, \widetilde{e}_2, \cdots, \widetilde{e}_2, \cdots, \widetilde{e}_M, \cdots, \widetilde{e}_M]^T$.
- For each I, $1 \leqslant I \leqslant M$, there are n $\widetilde{e}_I$'s in $\mathbf{e}_2$.
- By definition, $\mathrm{G} \cdot \mathbf{e}_2 = \lambda_2 \mathbf{e}_2$ and $\mathbf{e}_2 \cdot \mathbf{e}_2 = 1$.

Components of eigenvector e_2



Theory (2)

- $\lambda_2 = \mathbf{e}_2^T \cdot \mathbf{G} \cdot \mathbf{e}_2 = \sum_{i,j=1}^{N} e_{2i} G_{ij} e_{2j}$, where e_{2i} is the ith component of $\mathbf{e}_2$.
- We have

$$\begin{aligned}\lambda_2 &= \sum_{i=1}^{N} e_{2i}\{G_{i1}\widetilde{e}_1 + G_{i2}\widetilde{e}_1 + \cdots + G_{in}\widetilde{e}_1 \\ &\quad + G_{in+1}\widetilde{e}_2 + \cdots + G_{iN}\widetilde{e}_M\}.\end{aligned}$$

Theory (3)

- If i and j belong to the same cluster, G_{ij} equals $-1/k_i$ with probability p_s and 0 with probability $1 - p_s$.
- If i and j belong to different clusters, G_{ij} equals $-1/k_i$ with probability p_l and 0 with probability $1 - p_l$. Thus

$$
\begin{aligned}
\lambda_2 \quad = \quad & \sum_{i=1}^{N} e_{2i}\{-n\frac{p_l}{k_i}\widetilde{e}_1 - n\frac{p_l}{k_i}\widetilde{e}_2 + \cdots \\
& +\widetilde{e}_I - n\frac{p_s}{k_i}\widetilde{e}_I - \cdots - n\frac{p_l}{k_i}\widetilde{e}_M\},
\end{aligned}
$$

where $\widetilde{e}_I$ is the value corresponding to the cluster that contains node i.

Theory (4)

- For random subnetworks, the degree distribution is centered about $k = np_s + (N-n)p_l \rightarrow k_i \approx k$.

$$\begin{aligned}\lambda_2 &\approx \sum_{I=1}^{M} n\widetilde{e}_I\{N\frac{p_l}{k}\widetilde{e}_I - n\frac{p_l}{k}\sum_{J=1}^{M}\widetilde{e}_J\} \\ &= N\frac{p_l}{k}\sum_{I=1}^{M} n\widetilde{e}_I^2 - (n\sum_{J=1}^{M}\widetilde{e}_J)^2\frac{p_l}{k}.\end{aligned}$$

- Since $\sum_{I=1}^{M} n\widetilde{e}_I^2 \approx \sum_{i=1}^{N} e_{2i}^2 = 1$, and $n\sum_{J=1}^{M}\widetilde{e}_J = \sum_{i=1}^{N} e_{2i}$, we have

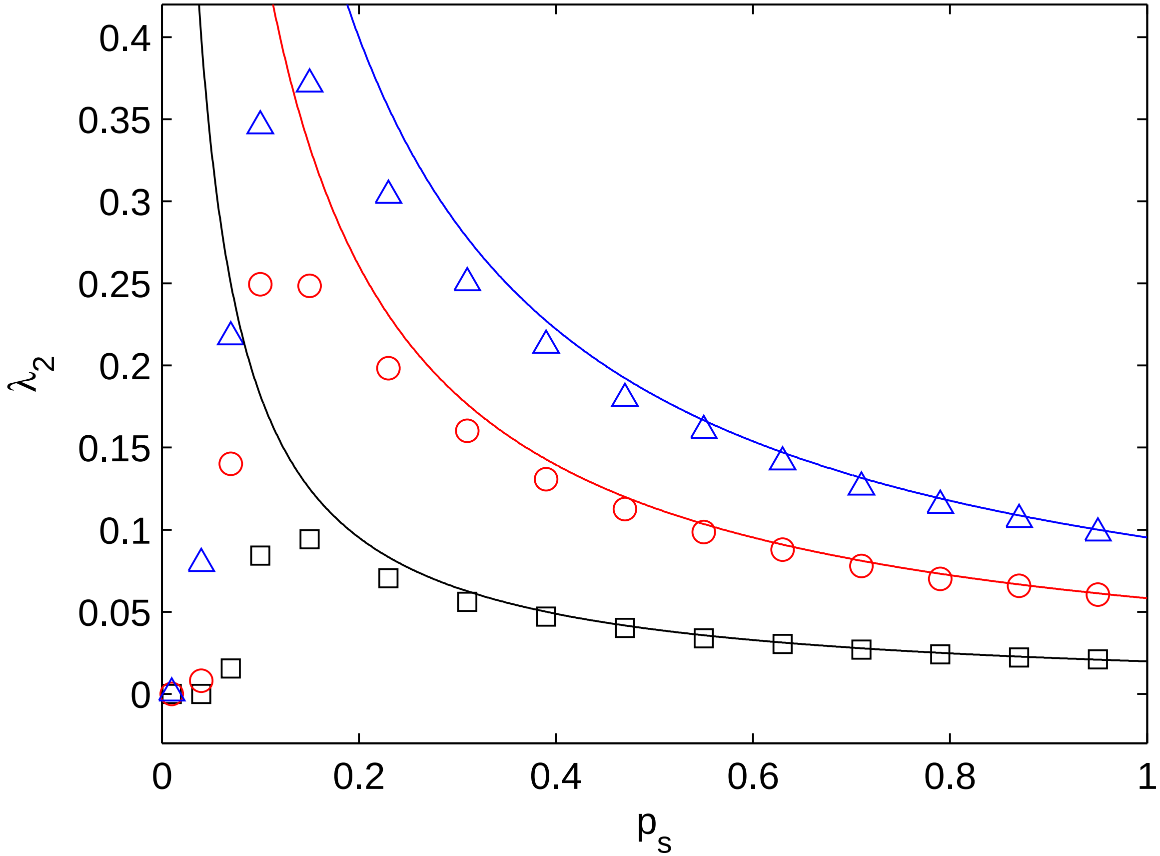
$$\lambda_2 = \frac{Np_l}{np_s + (N-n)p_l} - (\sum_{i=1}^{N} e_{2i})^2\frac{p_l}{k}.$$

Theory (5)

- $\lambda_2 = Np_l/[np_s + (N-n)p_l] - (\sum_{i=1}^{N} e_{2i})^2$
- The normalized eigenvector $\mathbf{e}_1$ of λ_1 is associated with the synchronized state, thus its components have constant values: $\mathbf{e}_1 = [1/\sqrt{N}, \cdots, 1/\sqrt{N}]^T$.
- If G is symmetric, then eigenvectors for different eigenvalues are orthogonal: $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, indicating $\sum_{l=1}^{N} e_{2l} = 0$.
- If G is slightly asymmetric, we expect $\sum_{i=1}^{N} e_{2i}$ to be small - safe to neglect the second term in λ_2. We have

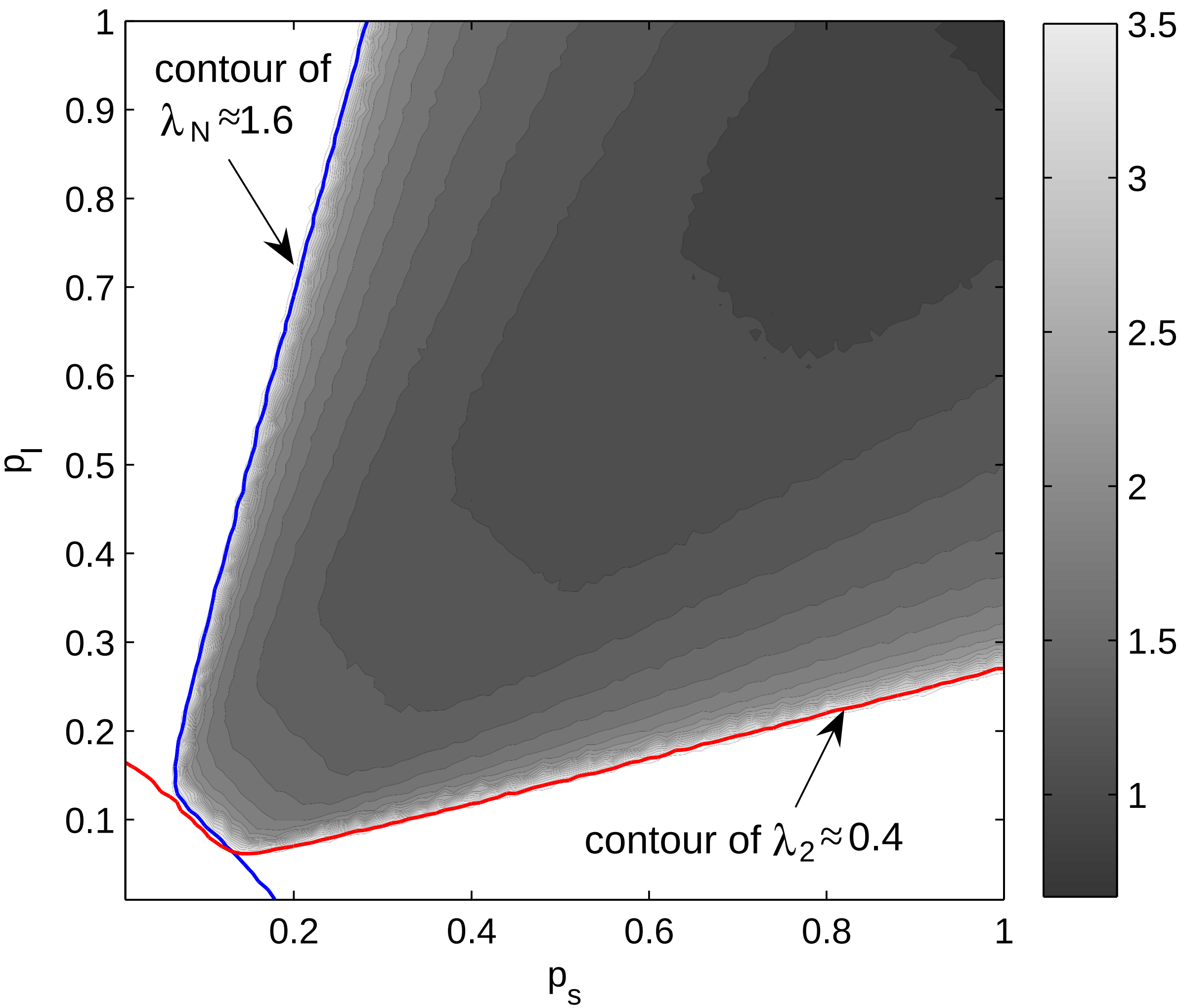
$$\lambda_2 \approx \frac{Np_l}{np_s + (N-n)p_l}$$

Numerical Verification



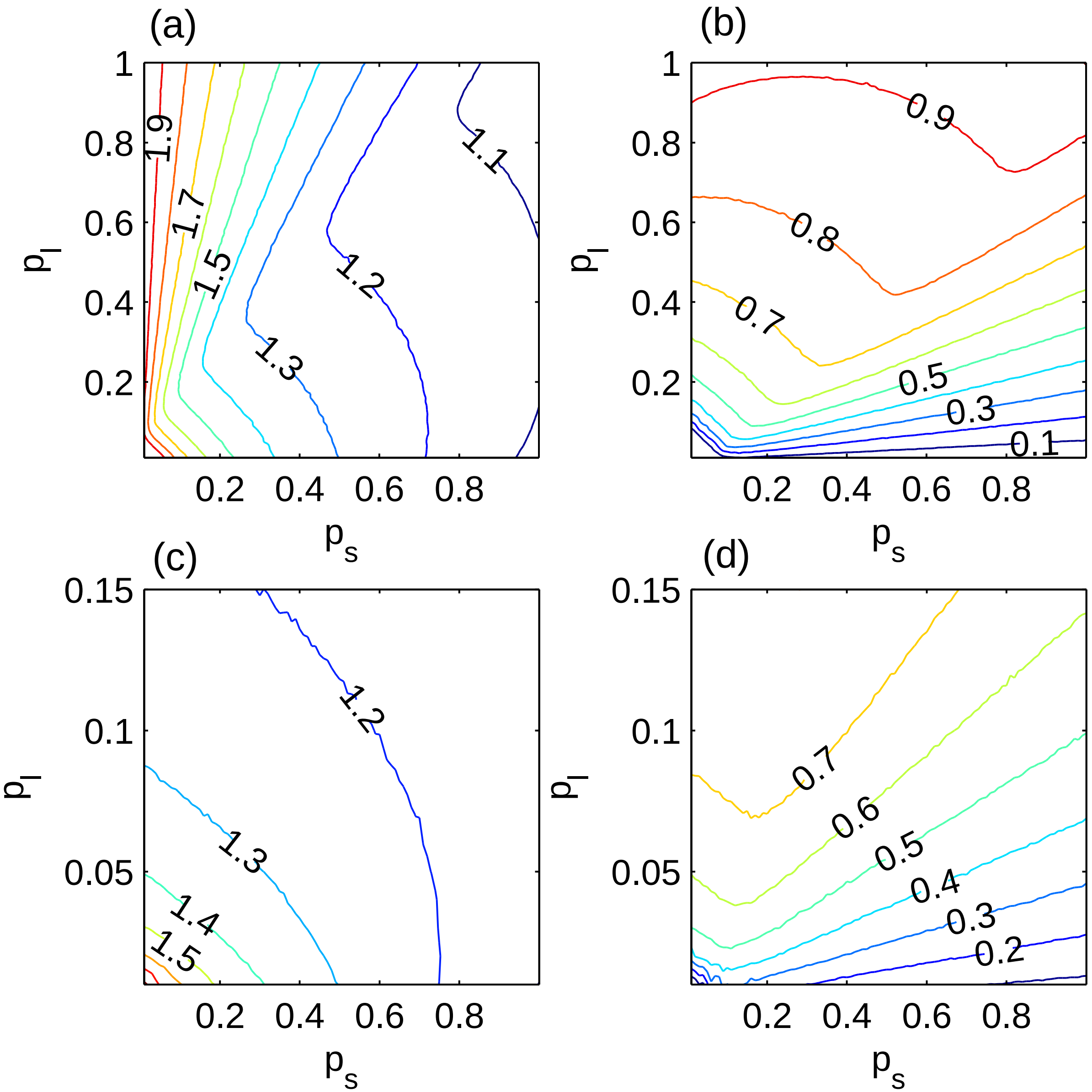
- $p_l = 0.01$ (squares), $p_l = 0.03$ (circles), and $p_l = 0.05$ (triangles); $N = 100$, $n = 50$, and 100 network realizations.

Clustered Network of Chaotic Logistic Maps



- $N = 100$, $M = 2$; chaotic logistic map on each node.

Contour plots of λ_N and λ_2



Conclusion

- Optimization: network is most synchronizable when the numbers of the inter-cluster and intra-cluster links are approximately equal.

Potential use:

- Insights into synchronization-dependent biological functions;
- Synchronous timing in large computer networks;
- Better coordination in transportation networks;
- . . .