Synchronization in Complex Networks

Ying-Cheng Lai

Department of Electrical Engineering, Department of Physics, Arizona State University
Main collaborators

- Frank Hoppensteadt, then ASU now NYU
- Adilson Motter, then ASU now Northwestern U
- Takashi Nishikawa, then ASU now Clarkson U
- Liang Huang, ASU
- Robert A. Gatenby, U. Arizona
Network Synchronization


- Synchronization in complex networks - highly relevant to biological functions, large-scale parallel processing, coordination in technological systems, etc.
Organizing biological information using the network idea has been fundamental to utilizing various systems-level approaches to understanding biological functions.

**Synchronizability** of oscillator networks, such as the networks of cortical neurons, is thought to play an important role for the functionality of the network.

Complex topology is common in biological networks.


Oscillator-Network Approach

\[ \dot{x}_i = F(x_i) - \varepsilon \sum_{j=1}^{N} G_{ij} H(x_j) \]

- coupling matrix, e.g.,

\[ G_{ij} = \begin{cases} 
-1/k_i, & \text{if nodes } i \text{ and } j \text{ interact with each other} \\
1, & \text{if } i = j \\
0, & \text{otherwise,} 
\end{cases} \]

\[ \sum_{j=1}^{N} G_{ij} = 0 \text{ for all } i \]

- How to choose $G$ to achieve stronger network synchronizability has been an active topic of research.
For any solution $s(t) \in \mathbb{R}^n$ of $\dot{x} = F(x)$ (periodic, chaotic, or else), $x_i(t) = s(t)$ for all $i$ is a synchronized solution for the whole oscillator network.

Stability of the solution is determined by linearizing and diagonalizing the equations, after which it takes the form

$$\dot{y}_i = \left[ DF(s(t)) - \varepsilon \lambda_i DH(s(t)) \right] y_i, \quad i=1,...,N$$

where $\lambda_1 = 0 \leq \lambda_2 \leq \ldots \leq \lambda_N$ are the eigenvalues of $L$.

$\lambda_1 = 0$ corresponds to the synchronization manifold:

$$\dot{y}_i = \left[DF(s(t))\right]y_i,$$

where $DF$ is the Jacobian matrix of node dynamics.

$0 \leq \lambda_2 \leq \ldots \leq \lambda_N$ correspond to $(N - 1)$-dimensional transverse subspace. Let $K \equiv \epsilon \lambda_i$ be the normalized coupling parameter. Transverse stabilities are determined by

$$\dot{y}_i = \left[DF(s(t)) - KDH(s(t))\right]y_i, \; i=2,\ldots,N$$
The largest Lyapunov exponent $\Psi(K)$ (= the Floquet exponent, if periodic) for the system

$$\dot{y} = \left[ DF(s(t)) - KDH(s(t)) \right] y$$

is called the master stability function. Pecora and Carroll (1998).

Physically realizable synchronization requires $\Psi(K) < 0$.

If $\Psi(K) < 0$ for $K_1 < K < K_2$, since $K \equiv \varepsilon \lambda_i$ ($i = 2, \ldots, N$), network synchronization occurs if $\varepsilon \lambda_2 > K_1$ and $\varepsilon \lambda_N < K_2$. 
Example of Master-Stability Function

- Rössler oscillator:
  \[ \mathbf{F}(\mathbf{x}) = [-y - z, x + 0.2y, 0.2 + z(x - 9)]^T. \]
- \( K_1 \approx 0.2 \) and \( K_2 \approx 4.6. \) Say \( \varepsilon = 0.5. \) Network synchronization requires \( \lambda_2 > 0.4 \) and \( \lambda_N < 9.2. \)
Question raised by Lou Pecora at the September 07 Aberdeen Chaos Conference.

Recent work at ASU: analysis and computation of MSFs for known chaotic oscillators.

Systems studied so far: Rössler oscillator, Lorenz oscillator, Chua’s circuit, Chen’s oscillator, Duffing’s oscillator, and van der Pol oscillator.

For each system, examined all 9 common and experimentally implementable coupling configurations.
Results: there always exists a coupling configuration for which the MSF is negative in a finite parameter interval, regardless of system details.

Synchronization in Scale-Free Networks

- Smaller average network distance generally does not imply enhanced synchronizability.
- Heterogeneity in the degree distribution is an obstacle for synchronization. Thus, some degree of homogeneity is expected in networks for which synchronizability is essential (e.g., neuronal networks) even if network distance is not optimal.
- Heterogeneity of load distribution appears to be responsible for the counter-intuitive effect.

[202 - Google Scholar]
Microscopic image of the colon (crypts);
Local communication subnetworks based on gap-junction conductance coupling;
Long-range diffusive interactions based on specific cell-membrane receptors.
Complex Clustered Networks in Biology

- **Metabolic networks**


- **Protein-protein interaction networks**

Clustered Networked Systems in Engineering

Figure 10

Clusters and massively parallel machines: Earth Simulator, Blue Gene/L, and ASCI Purple.
Clustered Topology in Social Networks

Complex clustered Networks

Local connection  Nodes
Long range links
Optimizing Synchronization

Two key parameters of interest:
1. $p_l$ - probability of inter-cluster links;
2. $p_s$ - probability of intra-cluster links.

In the two-dimensional parameter plane defined by $0 \leq (p_s, p_l) \leq 1$, where can the network be best synchronized?

Intuition: as $p_l$ or $p_s$ is increased, there are more links in the network, so it should be easier to achieve synchronization.
Optimal synchronizability occurs for $p_l \approx p_s$. For small $p_l$, increasing $p_s$ can lead to destruction of synchronization.


Network synchronization is meaningful in a probabilistic sense.

Calculate “synchronization width” defined by

\[ W(t) = \langle |x(t) - \langle x(t) \rangle| \rangle, \]

where \( \langle \cdot \rangle \) is average over nodes in the network.

Define \( P_{syn} \) - probability for \( W(t) < \delta \) (small number, say 0.01) for a large time period.

- \( P_{syn} \approx 0 \) - network is unsynchronizable;
- \( P_{syn} > 0 \) - network is synchronizable;
- \( P_{syn} \approx 1 \) - network is strongly synchronizable.
Classical Rössler oscillator on each node; \( N = 100 \), \( M = 2 \), and \( \varepsilon = 0.2 \).
Contour Plots of $\lambda_N$ and $\lambda_2$

- Reminder: synchronizable if $\lambda_2 > 0.4$ and $\lambda_N < 9.2$.
- Key: $\lambda_2$. For fixed $p_l$ (small), $\lambda_2(p_s)$ is a decreasing function for $p_s$ large.
For a clustered network, the components of the eigenvector $e_2$ have approximately the same value within any cluster, while they can be quite different for different clusters.

So write

$$e_2 \approx [\tilde{e}_1, \cdots, \tilde{e}_1, \tilde{e}_2, \cdots, \tilde{e}_2, \cdots, \tilde{e}_M, \cdots, \tilde{e}_M]^T.$$ 

For each $I$, $1 \leq I \leq M$, there are $n \tilde{e}_I$’s in $e_2$.

By definition, $G \cdot e_2 = \lambda_2 e_2$ and $e_2 \cdot e_2 = 1$. 
Components of eigenvector $e_2$
\[ \lambda_2 = e_2^T \cdot G \cdot e_2 = \sum_{i,j=1}^{N} e_{2i} G_{ij} e_{2j}, \text{ where } e_{2i} \text{ is the } i\text{th component of } e_2. \]

We have

\[
\lambda_2 = \sum_{i=1}^{N} e_{2i} \{ G_{i1} \tilde{e}_1 + G_{i2} \tilde{e}_1 + \cdots + G_{in} \tilde{e}_1 \\
+ G_{in+1} \tilde{e}_2 + \cdots + G_{iN} \tilde{e}_M \}. 
\]
If $i$ and $j$ belong to the same cluster, $G_{ij}$ equals $-1/k_i$ with probability $p_s$ and 0 with probability $1 - p_s$.

If $i$ and $j$ belong to different clusters, $G_{ij}$ equals $-1/k_i$ with probability $p_l$ and 0 with probability $1 - p_l$. Thus

$$
\lambda_2 = \sum_{i=1}^{N} e_{2i} \{-n \frac{p_l}{k_i} \tilde{e}_1 - n \frac{p_l}{k_i} \tilde{e}_2 + \cdots + \tilde{e}_I - n \frac{p_s}{k_i} \tilde{e}_I - \cdots - n \frac{p_l}{k_i} \tilde{e}_M\},
$$

where $\tilde{e}_I$ is the value corresponding to the cluster that contains node $i$. 
For random subnetworks, the degree distribution is centered about $k = np_s + (N - n)p_l \rightarrow k_i \approx k$.

$$\lambda_2 \approx \sum_{I=1}^{M} n\tilde{e}_I \left\{ N\frac{p_l}{k}\tilde{e}_I - n\frac{p_l}{k} \sum_{J=1}^{M} \tilde{e}_J \right\}$$

$$= N\frac{p_l}{k} \sum_{I=1}^{M} n\tilde{e}_I^2 - \left( n\sum_{J=1}^{M} \tilde{e}_J \right)^2 \frac{p_l}{k}.$$ 

Since $\sum_{I=1}^{M} n\tilde{e}_I^2 \approx \sum_{i=1}^{N} e_{2i}^2 = 1$, and $n \sum_{J=1}^{M} \tilde{e}_J = \sum_{i=1}^{N} e_{2i}$, we have

$$\lambda_2 = \frac{Np_l}{np_s + (N - n)p_l} - \left( \sum_{i=1}^{N} e_{2i} \right)^2 \frac{p_l}{k}.$$
\[ \lambda_2 = N p_l / [np_s + (N - n)p_l] - (\sum_{i=1}^{N} e_{2i})^2 \]

The normalized eigenvector \( e_1 \) of \( \lambda_1 \) is associated with the synchronized state, thus its components have constant values: \( e_1 = [1/\sqrt{N}, \cdots, 1/\sqrt{N}]^T \).

If \( G \) is symmetric, then eigenvectors for different eigenvalues are orthogonal: \( e_i \cdot e_j = \delta_{ij} \), indicating \( \sum_{i=1}^{N} e_{2i} = 0 \).

If \( G \) is slightly asymmetric, we expect \( \sum_{i=1}^{N} e_{2i} \) to be small - safe to neglect the second term in \( \lambda_2 \). We have

\[ \lambda_2 \approx \frac{N p_l}{np_s + (N - n)p_l} \]
$p_l = 0.01$ (squares), $p_l = 0.03$ (circles), and $p_l = 0.05$ (triangles); $N = 100$, $n = 50$, and 100 network realizations.
Clustered Network of Chaotic Logistic Maps

- $N = 100$, $M = 2$; chaotic logistic map on each node.
Contour plots of $\lambda_N$ and $\lambda_2$
**Optimization**: network is most synchronizable when the numbers of the inter-cluster and intra-cluster links are approximately equal.

**Potential use**: 
- Insights into synchronization-dependent biological functions;
- Synchronous timing in large computer networks;
- Better coordination in transportation networks;
- ...